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p-Steinberg Characters of Finite Simple Groups

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Let G be a finite group and p a prime divisor of $|G|$. A p-Steinberg character of G is an irreducible character χ of G such that $\chi(x) = \pm|C_G(x)|_p$ for every p'-element $x \in G$. A conjecture of W. Feit states that if a finite simple group G has a p-Steinberg character then G is a finite simple group of Lie type in characteristic p. In this paper we prove this conjecture, using the classification of finite simple groups. © 1997 Academic Press

1. INTRODUCTION

Let G be a finite group and p a prime. Following [3], we call any irreducible character χ of G such that $\chi(x) = \pm|C_G(x)|_p$ for every p'-element $x \in G$ a *p-Steinberg character* of G. Here N_p denotes the p-part of an integer N. As mentioned in [3], if χ is a p-Steinberg character, then χ has p-defect 0 and so $\chi(y) = 0$ for every p-singular element $y \in G$. If p does not divide $|G|$, then p-Steinberg characters are precisely the linear characters of G which accept only values ± 1. Therefore it is natural to make the assumption that p is a prime divisor of $|G|$.

If G is a finite group of Lie type in characteristic p, then the Steinberg character St is obviously a p-Steinberg character. In [3] W. Feit stated the following conjecture:

Conjecture 1.1. Let G be a finite simple group and p a prime divisor of $|G|$. Suppose G has a p-Steinberg character. Then G is a finite simple group of Lie type in characteristic p.

Feit also suggested that using the classification of finite simple groups one should be able to prove Conjecture 1.1.

The aim of this paper is to prove the above conjecture of Feit.

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THEOREM 1.2. *Let G be a finite simple group and p a prime divisor of $|G|$. Suppose G has a p-Steinberg character. Then G is isomorphic to a finite simple group of Lie type in characteristic p.*

The cases where G is an alternating group $\mathbb{A}_n$ or $PSL_n(q)$ and where $|G|_p = p$ have been treated in [2] and in [3], respectively.

By Theorem A [3], a given finite simple group can have at most one p-Steinberg character. Thus Theorem 1.2 yields immediately the following consequence:

COROLLARY 1.3. *Let G be a finite simple group and p a prime divisor of $|G|$. Suppose G has a p-Steinberg character χ. Then G is isomorphic to a finite simple group of Lie type in characteristic p, and χ is then the Steinberg character St of G.*

Observe that the condition $\chi(1) = |G|_p$ is fairly restrictive. Namely, let G be a finite simple group of Lie type in characteristic $r \neq p$. Then $|G|_p$ is relatively small and it is quite close to the Landázuri–Seitz–Zalesskii bound $l(G)$ (for the minimum degree of nontrivial projective representations of G in characteristic other than r, cf. [4, 5]). Furthermore, it is proved in [6] that, if G is a finite classical simple group and $\chi \in \mathrm{Irr}(G)$, then either χ is trivial, or a *Weil character*, or $\chi(1)$ is at least, roughly speaking, $(l(G))^{3/2}$. Using this result for the classical groups and the results of [4, 5] for the exceptional groups, one could prove the following strengthening of Conjecture 1.1.

Let G be a finite simple group of Lie type defined over $\mathbb{F}_q$, q a prime power and $q \geq 9$, and let $G \not\simeq PSL_2(q)$. Suppose G has an irreducible character χ whose degree is a power $p^m > 1$ of a prime p. Then one of the following holds.

(i) *p divides q.*

(ii) *G is of type A_n, ${}^2A_n$, or C_n, and χ is a Weil character.*

However, we are not going to prove this statement here. (The reason why we have to exclude the groups $PSL_2(q)$ is the following: all nontrivial characters of $PSL_2(q)$ are in some sense Weil characters. Namely, if we view $G := PSL_2(q)$ as $PSL_2(q)$ as usual, then the Weil characters of G are the ones of degree q, $q+1$ (and $(q+1)/2$ if q is odd). If we view G as $PSU_2(q)$, then the Weil characters of G are the ones of degree q, $q-1$ (and $(q-1)/2$ if q is odd). Finally, if q is odd and we view G as $PSp_2(q)$, then the Weil characters of G are the ones of degree $(q \pm 1)/2$.)

Throughout the paper, $d(G)$ is the minimum degree of nontrivial projective complex representations of a finite simple group G, $\nu_p(N)$ is the p-adic valuation of an integer N, E_n is the identity $n \times n$-matrix (over any field), $\mathbb{F}_q^{\cdot} = \mathbb{F}_q \setminus \{0\}$.

2. PRELIMINARIES

We start with a well-known arithmetical lemma

LEMMA 2.1. *Let p be a prime, $m \in \mathbb{N}, q \in \mathbb{Z}$ an integer coprime to p. Then the following hold.*

(i) $\nu_p(q^m - 1) = \nu_p(q - 1) + \nu_p(m)$ *if p is odd and divides $q - 1$, or if $p = 2$ and 4 divides $q - 1$.*

(ii) *If 4 divides $q + 1$ then $\nu_2(q^m - 1)$ is equal to 1 if m is odd and $\nu_2(q + 1) + \nu_2(m)$ if m is even.*

LEMMA 2.2. *Suppose p, q are primes such that $p^a - q^b = 1$ for some integers $a, b \geq 1$. Then $(p^a, q^b) = (3^2, 2^3), (p, 2^b), (2^a, q)$.*

Proof. If $a = 1$, then $p = q^b + 1 \geq 3$, whence $q = 2$. Next suppose $a > 1$ and q is odd. Then $p = 2$ and $q^b + 1 = 2^a$ is divisible by 4. Therefore b is odd. If $b \geq 3$ then 2^a is divisible by $c := (q^b + 1)/(q + 1)$ $(\equiv 1 \bmod 2)$ and $c > 1$, a contradiction. Hence $b = 1$. Finally, suppose that $a > 1$ and $q = 2$. By the same reason as above a cannot have an odd divisor greater than 1, whence $a = 2^d$. If $d > 1$, then 2^b is divisible by $e := (p^{2^{d-1}} + 1)/2 \equiv 1 \pmod 2$ with $e > 1$, again a contradiction. Hence $d = 1$, $a = 2$, $2^b = p^2 - 1$, yielding $p = 3$, $q^b = 2^3$. ∎

LEMMA 2.3. *Let $n \geq 2, q$ a prime power, and p a prime not dividing q. Then one of the following holds.*

(i) $\mathcal{Q}_1 := ((q^{2n} - 1)/\gcd(2, q - 1))_p < q^n - 1$.

(ii) $2 \mid q, p = q^n \pm 1$.

(iii) $(n, q, p) = (3, 2, 3), (2, 3, 2)$.

Proof. It is clear that $\mathcal{Q}_1 = (q^n \pm 1)_p$. Hence, if none of $q^n \pm 1$ is a p-power, then $\mathcal{Q}_1 \leq (q^n + 1)/2 < q^n - 1$. Suppose $q^n \pm 1 = p^a$. Then by Lemma 2.2 we arrive at (ii) or (iii). ∎

LEMMA 2.4. *Let $n \geq 3$, q a prime power, and p a prime not dividing q. Then one of the following holds.*

(i) $\mathcal{Q}_2 := ((q^{2n-2} - 1)(q^{2n} - 1)/(q^2 - 1))_p < q^{n-1} - 1$.

(ii) $p \mid (q^n \pm 1)$ *and* $o_p(q) = n, 2n$, *where $o_p(q)$ is the order of q in $\mathbb{F}_p^{\cdot}$.*

(iii) $2 \mid q, p = q^{n-1} \pm 1$.

(iv) $(n, q, p) = (3, 2, 3), (3, 3, 2), (4, 2, 3), (4, 3, 2)$.

Proof. We may suppose that $\mathscr{Q}_2 > 1$. First we consider the case where $p \mid (q^2-1)$. Observe that when $p=2$ we have $4 \mid (q^2-1)$. Hence $\mathscr{Q}_2 = (q^2-1)_p \cdot p^{\nu_p(n-1)+\nu_p(n)}$ by Lemma 2.1. In particular, $\mathscr{Q}_2 \leq n(q^2-1)$. If $n=5$, then $q^4-1 \geq 5(q^2-1) \geq \mathscr{Q}_2$. The equality can occur only when $q=2$, but in this case $p=3$ and so $\mathscr{Q}_2 = 3 < q^4-1$. If $n>5$ then $q^{n-1}-1 > n(q^2-1) \geq \mathscr{Q}_2$. If $n=4$ and $q \geq 4$, then $q^3-1 > 4(q^2-1) \geq \mathscr{Q}_2$. If $n=4$ and $q=2,3$ then we arrive at (iv). Now suppose that $n=3$. If $p \geq 5$ then $\mathscr{Q}_2 = (q^2-1)_p < q^2-1$. If $p=3$ and $q>2$ then $\mathscr{Q}_2 = 3(q^2-1)_3 < q^2-1$. If $p=2$ and $q>3$ then $\mathscr{Q}_2 = 2(q^2-1)_2 < q^2-1$. The exceptions $(n,q,p) = (3,2,3),(3,3,2)$ have been listed in (iv).

Next we consider the case where p divides $q^{2(n-1)}-1$ but not q^2-1. In particular $p \geq 5$. Clearly, $\mathscr{Q}_2 = (q^{n-1} \pm 1)_p$. If $q^{n-1} \pm 1$ is not a p-power, then $\mathscr{Q}_2 \leq (q^{n-1}+1)/2 < q^{n-1}-1$. If $q^{n-1} \pm 1 = p^a$, then by Lemma 2.2 we arrive at (iii).

Finally, let p divide $q^{2n}-1$ but not q^2-1. In particular $p \geq 5$. Denoting $l = o_p(q)$, we see that $l \geq 3$ and $l \mid 2n$. By Lemma 2.1, $\mathscr{Q}_2 = (q^{2n}-1)_p = (q^l-1)_p \cdot p^{\nu_p(2n/l)} \leq 2n(q^l-1)/l$. If $l \geq n$ we arrive at (ii). If $l = 2n/3$ or $2n/4$, then $\nu_p(2n/l) = 0$ whence $\mathscr{Q}_2 = (q^l-1)_p \leq (q^{2n/3}-1)/2 < q^{n-1}-1$ by Lemmas 2.1 and 2.2. If $3 \leq l \leq 2n/5$, then $q^{n-1}-1 > 2n(q^l-1)/l \geq \mathscr{Q}_2$. ∎

COROLLARY 2.5. *Let $n \geq 3$, Q an integer such that $q := |Q|$ is a prime power, and p a prime which does not divide q. Then one of the following holds.*

(a) $\mathscr{Q}_3 := |(Q^{n-1}-1)(Q^n-1)/(Q-1)|_p < q^{n-1}-1$.

(b) *$Q = q$ and $o_p(q) = n$.*

(c) $Q = q = 2$, $p = 2^{n-1}-1$.

(d) $(n, Q, p) = (3, \pm 2, 3), (3, \pm 3, 2), (4, -2, 3)$. ∎

Proof. It suffices to consider the case where $\mathscr{Q}_3 > 1$. It is clear that $\mathscr{Q}_3$ divides $\mathscr{Q}_2$. Hence we can apply Lemma 2.4. In case (i) of this lemma we are done. Consider case (ii). If $Q = q$ we come to conclusion (b). If $Q = -q$ then

$$\mathscr{Q}_3 = |Q^n-1|_p \leq \left|\frac{Q^n-1}{Q-1}\right| = \frac{q^n \pm 1}{q+1} < q^{n-1}-1.$$

Next consider case (iii). If $Q = q$ we get (c) or (d). If $Q = -q$ then we come to (d). Finally, (iv) leads to the possibilities listed in (d). ∎

The case of quasi-simple groups reduces to the simple case by means of the following easy observation:

LEMMA 2.6. *Let a finite perfect group G have a faithful p-Steinberg character χ. Then $Z(G) = 1$.*

Proof. By Schur's Lemma, $Z(G)$ is cyclic. Suppose that $m := |Z(G)| > 1$. Let z be a generator of $Z(G)$. Then $\chi(z) = \varepsilon\chi(1)$ for an mth primitive root of unity. In particular, z cannot be p-singular, i.e., m is coprime to p. But in this case $\varepsilon\chi(1)$ should be an integer; therefore $\varepsilon = -1$ and $m = 2$. This implies p is odd, and $\chi(1) = |G|_p$ is odd. If Φ denotes a representation affording the character χ, then $\det(\Phi(z)) = -1$. On the other hand, the perfectness of G implies that $\det(\Phi(g)) = 1$ for any $g \in G$, a contradiction. ∎

Our analysis is heavily based on considering some *groups of extraspecial type*, i.e., an r-group R (r a prime) such that $T := [R, R] = Z(R) \neq 1$ and $[x, R] = T$ for any $x \in R \setminus T$. Let $\chi \in \mathrm{Irr}(R)$ be of degree > 1. Then there is a $z \in T$ such that $z \notin \mathrm{Ker}\chi$. Let $x \in R \setminus T$. Then we can find $y \in R$ such that $[x, y] = z$. This forces $\chi(x) = 0$. On the other hand, $\chi|_T = \chi(1)\rho$ for some $\rho \in \mathrm{Irr}(T) \setminus \{1_T\}$. We have just shown that R has precisely $|R/T|$ linear characters and $|T| - 1$ irreducible characters of degree $\sqrt{|R/T|}$.

LEMMA 2.7. *Let G be a finite group having a p-Steinberg character χ. Suppose G has an r-subgroup R of extraspecial type, where r is a prime and $r \neq p$. Suppose in addition that the following conditions hold.*

(i) *There is an integer D such that $D = (G : C_G(t))_p$ for all $t \in T \setminus \{1\}$.*

(ii) *If $1 \neq t \in T := Z(R)$ then $|t^G \cap T|$ is equal to $|T| - 1$ or $(|T| - 1)/2$.*

Then there are an integer B and an integer $C = \pm D$ such that $\chi(1) = BC$ and $\chi(t) = B$ for all t, $1 \neq t \in T$. Moreover, $C - 1$ is divisible by $\sqrt{|R| \cdot |T|}$.

Proof. Denote $A = \chi(1)$. First we show that there are an integer B and an integer $C = \pm D$ such that $A = BC$ and $\chi(t) = B$ for all $t, 1 \neq t \in T$. It is so if $\chi(t)$ is the same for all $t \in T \setminus \{1\}$. If not, then by (i) and (ii) we see that $\chi(t) = B = A/D$ for $(|T| - 1)/2$ nontrivial elements $t \in T$ and $\chi(t) = -B$ for the remaining $(|T| - 1)/2$ elements. In this case

$$(\chi|_T, 1_T)_T = \frac{1}{|T|}\left(A + \frac{|T| - 1}{2}(B - B)\right) = \frac{A}{|T|};$$

hence $|T|$ divides A. But $1 \neq |T|$ is an r-power, A is a p-power, and $r \neq p$, a contradiction.

Now we consider any $\rho \in \mathrm{Irr}(T)\backslash\{1_T\}$. Then

$$(\chi|_T, \rho)_T = \frac{1}{|T|}\left(A + B\sum_{1\neq t\in T}\rho(t)\right) = \frac{A-B}{|T|} + B(1_T, \rho)_T = \frac{A-B}{|T|}.$$

On the other hand, due to the above description of the irreducible characters of R we have

$$\chi|_R = \sum_{\alpha\in\mathrm{Irr}(R),\ \alpha(1)=1} m_\alpha \alpha + \sum_{\alpha\in\mathrm{Irr}(R),\ \alpha|_T = N\rho} n_\rho \alpha.$$

The second sum runs over all $\rho \in \mathrm{Irr}(T)\backslash\{1_T\}$; furthermore, $N = \sqrt{|R/T|}$. This yields

$$\chi|_T = \left(\sum m_\alpha\right)\cdot 1_T + N\sum n_\rho \rho.$$

Thus $(A-B)/|T| = Nn_\rho$. In particular, $B(C-1) = A - B$ is divisible by $M := N\cdot|T| = \sqrt{|R|\cdot|T|}$. Since M is an r-power and B is a p-power, we come to the conclusion that $C-1$ is divisible by M. ∎

Observe that conditions (i) and (ii) in Lemma 2.7 are automatically satisfied if $|T| = 2, 3$. Even more, one can show that the conclusion of Lemma 2.7 holds, if one replaces conditions (i) and (ii) by the condition that $|T| = r$.

3. SYMPLECTIC GROUPS

To illustrate our method, we prove Theorem 1.2 for the case of symplectic groups. Let $q = r^m$ be a power of a prime $r, n \geq 2$, $G = PSp_{2n}(q)'$. Suppose G has a p-Steinberg character χ, where p divides $|G|$ but not q. We want to show that this can happen only when $(n, q, p) = (2,2,3), (2,3,2)$, where G has a p-Steinberg character (because of the isomorphisms $Sp_4(2)' \simeq L_2(9)$ and $PSp_4(3) \simeq SU_4(2)$).

Using [1], one easily gets our statement for the groups $Sp_4(2)'$, $PSp_4(3)$, and $Sp_6(2)$. Henceforth we may suppose that $G = PSp_{2n}(q)$ with $(n, q) \neq (2,2), (2,3), (3,2)$.

(1) First we assume that q is odd. It is more convenient to work with $H = Sp_{2n}(q)$. Let $V = \mathbb{F}_q^{2n}$ denote the natural symplectic module for H, and let $P = St_H(\langle v\rangle_{\mathbb{F}_q})$ be the stabilizer of a point $\langle v\rangle_{\mathbb{F}_q}$, $0 \neq v \in V$. It is not difficult to show that $R := O_r(P)$ can be identified with the set $\{[X, Y, z] \mid X, Y \in \mathbb{F}_q^{n-1},\ z \in \mathbb{F}_q\}$ endowed with the following group operation

$$[X, Y, z]\circ[X', Y', z'] = \left[X + X', Y + Y', z + z' + \sum_{i=1}^{n-1}(x_i y_i' - x_i' y_i)\right],$$

if

$$X = (x_1, \ldots, x_{n-1}), \qquad Y = (y_1, \ldots, y_{n-1}),$$
$$X' = (x'_1, \ldots, x'_{n-1}), \qquad Y' = (y'_1, \ldots, y'_{n-1}).$$

Clearly, $[R, R] = Z(R)$ coincides with the subgroup $T = \{t_c := [0, 0, c] \mid c \in \mathbb{F}_q\}$. Moreover, $[x, R] = T$ for any $x \in R \setminus T$; hence R is of extraspecial type. If $c \neq 0$, then t_c is a transvection, with the matrix $\mathrm{diag}\left(\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right), E_{2n-2}\right)$ in a suitable symplectic basis of V. Therefore, $C_H(t_c) \geq \mathbb{Z}_{\gcd(2,q-1)} \times Sp_{2n-2}(q)$ (the first group is just the centre of $Sp_2(q)$). Observe that $h_a t_c h_a^{-1} = t_{a^2 c}$, if $h_a = \mathrm{diag}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), E_{2n-2}\right)$, $a \in \mathbb{F}_q^{\cdot}$. This means t_c and $t_{c'}$ are H-conjugate whenever $c, c' \in \mathbb{F}_q^{\cdot}$, and c'/c is a square in $\mathbb{F}_q$. Similarly, one easily checks that all the transvections t_c, $c \neq 0$, are conjugate in the conformal symplectic group $K = CSp_{2n}(q) = H \cdot \mathbb{Z}_{q-1}$.

Now we come back to G. Since $R \cap Z(H) = 1$ and $G = H/Z(H)$, we can view R as a subgroup of G. The above discussion shows that R satisfies all the conditions of Lemma 2.7. Moreover,

$$D \leq \left(H : \left(\mathbb{Z}_{\gcd(2,q-1)} \times Sp_{2n-2}(q)\right)\right)_p = \mathscr{Q}_1,$$

where $\mathscr{Q}_1$ has been defined in Lemma 2.3. Applying Lemma 2.7, we observe that $C \neq 1$ (otherwise $\mathrm{Ker}\,\chi$ would contain the transvections t_c, and so $\mathrm{Ker}\,\chi = G$, $\chi(1) = 1$, a contradiction). Hence, according to Lemma 2.7, $0 \neq C - 1$ is divisible by $\sqrt{|R| \cdot |T|} = q^n$. This implies $\mathscr{Q}_1 \geq D = |C| \geq q^n - 1$. By Lemma 2.3 and by our assumptions on (n, q), we must have $2 \mid q$ and $p = q^n \pm 1$. In this case it is easy to see that $\chi(1) = |G|_p = p$. Since $(n, q) \neq (2, 2), (3, 2)$, we have by Theorem 5.5 [6] that $d(G) = (q^n - 1)(q^n - q)/(2(q + 1))$, and so $1 < \chi(1) < d(G)$, a contradiction. (The desired contradiction for the final configuration that $|G|_p = p$ could also be deduced from Theorem 6.1 [3].)

(2) Now we deal with the case where q is even. Suppose $n \geq 3$. Embed $SO^+_{2n}(q)$ in $Sp_{2n}(q)$ and consider the group R of order $q^{1+2(2n-4)}$ constructed in Proposition 5.1 (below). (Thus, R is contained in $O_r(P)$, where P is the stabilizer in G of an isotropic line.) Then R satisfies all the conditions of Lemma 2.7. Moreover, $C_G(t) \geq SL_2(q) \times Sp_{2n-4}(q)$ for any nontrivial element $t \in T := Z(R)$. This implies $D \leq \mathscr{Q}_2$, where $\mathscr{Q}_2$ has been defined in Lemma 2.4. Applying Lemma 2.7, we observe that $C \neq 1$ (otherwise $\mathrm{Ker}\,\chi$ would contain all the elements $t \in T$, and so $\mathrm{Ker}\,\chi = G$, $\chi(1) = 1$, a contradiction). Hence, According to Lemma 2.7, $0 \neq C - 1$ is divisible by $\sqrt{|R| \cdot |T|} = q^{2n-3}$. This implies $\mathscr{Q}_2 \geq D = |C| \geq q^{2n-3} - 1$. Since $n \geq 3$, q even and $(n, q) \neq (3, 2)$, we easily obtain that $\mathscr{Q}_2 < q^{2n-3} - 1$, a contradiction.

Finally, let $G = Sp_4(q)$ with q even and $q \geq 4$. Then $\chi(1)$ is equal to $\mathscr{Q}_5 := ((q^2 - 1)(q^4 - 1))_p$. On the other hand, the minimum degree of nontrivial characters of G is $q(q-1)^2/2$ (cf. [5], [6]). From this it follows that $\mathcal{Q}_5 \geq q(q-1)^2/2$. This can happen only when $q = 4$ and $p = 5$. This last configuration is excluded due to that $G = Sp_4(4)$ has no irreducible characters of degree equal to $\mathcal{Q}_5 = 25$.

4. UNITARY GROUPS AND SPECIAL LINEAR GROUPS

In this section we prove Theorem 1.2 for the case of unitary groups. Let $q = r^m$ be a power of a prime r, $n \geq 3$, $G = PSU_n(q)$. We except the case $(n, q) = (3, 2)$ since $U_3(2)$ is solvable. Suppose G has a p-Steinberg character χ, where p divides $|G|$ but not q. We want to show that this can happen only when $(n, q, p) = (4, 2, 3)$ where G has a 3-Steinberg character (because of the isomorphism $SU_4(2) \simeq PSp_4(3)$).

Using [1], one easily gets our statement for the groups $SU_3(3), SU_4(2)$. Henceforth we may suppose that $G = PSU_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2), (3, 3), (4, 2)$.

It is more convenient to work with $H = SU_n(q)$. Let $V = \mathbb{F}_q^n$ be the natural module for H, endowed with a Hermitian form with Gram matrix $\mathrm{diag}(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), E_{n-2})$ in a basis $(v_1, v_2, \ldots, v_n)$. Let $P = St_H(\langle v_1 \rangle_{\mathbb{F}_{q^2}})$ be the stabilizer of the isotropic point $\langle v_1 \rangle_{\mathbb{F}_{q^2}}$. It is not difficult to see that $R = O_r(P)$ can be identified with the set.

$$\left\{ [X, z] \mid X = (x_1, \ldots, x_{n-2}) \in \mathbb{F}_{q^2}^{n-2},\ z \in \mathbb{F}_{q^2},\ z + z^q + \sum_{k=1}^{n-2} x_k^{q+1} = 0 \right\}$$

endowed with the group operation

$$[X, z] \circ [X', z'] = \left[X + X', z + z' - \sum_{k=1}^{n-2} x_k^q x_k' \right],$$

if $X = (x_1, \ldots, x_{n-2})$, $X' = (x_1', \ldots, x_{n-2}')$. The centre and the commutator group of R coincide with each other and are equal to $T = \{t_c := [0, c] \mid c \in \mathbb{F}_{q^2}, c + c^q = 0\}$. Moreover, $[x, R] = T$ for any $x \in R \setminus T$; hence R is of extraspecial type. If $c \neq 0$, then t_c is a transvection, with the matrix $\mathrm{diag}(\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right), E_{n-2})$ in the basis $(v_1, v_2, \ldots, v_n)$ of V. Since $n \geq 3$, H contains the elements $h_a = \mathrm{diag}(a, a^{-q}, a^{q-1}, E_{n-3})$ (in the indicated basis of V), $a \in \mathbb{F}_{q^2}^{\cdot}$. Observe that $h_a t_c h_a^{-1} = t_{a^{q+1}c}$. This means that all the transvections t_c, $c \neq 0$, are conjugate in H. Also, $C_H(t_c)$ contains the subgroup

$$\left\{\mathrm{diag}(a, a, X) \mid a \in \mathbb{F}_{q^2},\, a^{q+1} = 1,\, X \in U_{n-2}(q),\, \det X = a^{-2}\right\}$$

isomorphic to $SU_{n-2}(q) \cdot \mathbb{Z}_{q+1}$.

Now we come back to G. Since $R \cap Z(H) = 1$ and $G = H/Z(H)$, we can view R as a subgroup of G. The above discussion shows that R enjoys all the conditions of Lemma 2.7. Moreover,

$$D \leq \left(H : \left(SU_{n-2}(q) \cdot \mathbb{Z}_{q+1}\right)\right)_p = \mathscr{Q}_3,$$

where $\mathscr{Q}_3$ is as defined in Corollary 2.5 and $Q = -q$. Applying Lemma 2.7, we observe that $C \neq 1$ (otherwise $\mathrm{Ker}\,\chi$ would contain the transvections t_c, and so $\mathrm{Ker}\,\chi = G$, $\chi(1) = 1$, a contradiction). Hence, according to Lemma 2.7, $0 \neq C - 1$ is divisible by $\sqrt{|R| \cdot |T|} = q^{n-1}$. This implies $\mathscr{Q}_3 \geq D = |C| \geq q^{n-1} - 1$. On the other hand, by Corollary 2.5 and by our assumptions on (n, q), we must have that $\mathscr{Q}_3 < q^{n-1} - 1$, a contradiction.

The case of $PSL_n(q)$ has already been dealt with in [2]. However, we want to give here another proof for this case. Let $q = r^m$ be a power of a prime $r, n \geq 2$, $G = PSL_n(q)$. Since the group $PSL_2(q)$ can easily be handled in a direct way, we may assume that $n \geq 3$. Suppose G has a p-Steinberg character χ, where p divides $|G|$ but not q. We want to show that this can happen only when $(n, q, p) = (3, 2, 7)$ where G has a 7-Steinberg character (because of the isomorphism $SL_3(2) \simeq PSL_2(7)$).

Using [1], one easily gets our statement for the groups $SL_3(2), SL_3(3)$, $SL_4(2)$. Henceforth we may suppose that $G = PSL_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2, (3, 3), (4, 2)$.

It is more convenient to work with $H = SL_n(q)$. Let $V = \mathbb{F}_q^n$ be the natural module for H, with a basis $(v_1, v_2, \dots, v_n)$. Setting $P = \mathrm{St}_H \langle v_2 \rangle_{\mathbb{F}_q}, \langle v_2, \dots, v_n \rangle_{\mathbb{F}_q})$, it is not difficult to see that $R = O_r(P)$ is a group of order q^{2n-3} of extraspecial type. Here $Z(R) = T = \{t_c \mid c \in \mathbb{F}_q\}$, and if $c \neq 0$ then t_c is a transvection, with the matrix $\mathrm{diag}(\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right), E_{n-2})$ in the given basis of V. Since $n \geq 3$, H contains the elements $h_a = \mathrm{diag}(a^{-1}, 1, a, E_{n-3})$ (in the chosen basis of V), $a \in \mathbb{F}_q^{\cdot}$. Observe that $h_a t_c h_a^{-1} = t_{ac}$. This means that all the transvections t_c, $c \neq 0$, are conjugate in H. Also, $C_H(t_c)$ contains the subgroup

$$\begin{aligned} &\left\{\mathrm{diag}(a, a, X) \mid a \in \mathbb{F}_q^{\cdot},\, X \in GL_{n-2}(q),\, \det X = a^{-2}\right\} \\ &\quad \simeq SL_{n-2}(q) \cdot \mathbb{Z}_{q-1}. \end{aligned}$$

Now we come back to G. Since $R \cap Z(H) = 1$ and $G = H/Z(H)$, we can view R as a subgroup of G. The above discussion shows that R satisfies all the conditions of Lemma 2.7. Moreover

$$D \leq \left(H : \left(SL_{n-2}(q) \cdot \mathbb{Z}_{q-1}\right)\right)_p = \mathscr{Q}_3,$$

where $\mathscr{Q}_3$ is as defined in Corollary 2.5 and $Q = q$. Applying Lemma 2.7, we observe that $C \neq 1$ (otherwise Ker χ would contain the transvections t_c, and so Ker $\chi = G$, $\chi(1) = 1$, a contradiction). Hence, according to Lemma 2.7, $0 \neq C - 1$ is divisible by $\sqrt{|R| \cdot |T|} = q^{n-1}$. This implies $\mathscr{Q}_3 \geq D = |C| \geq q^{n-1} - 1$. By Corollary 2.5 and by our assumptions on (n, q), we must have that either $q = 2$ and $p = 2^{n-1} - 1$, or $o_p(q) = n$. In the former case $\chi(1) = |G|_p = p$. On the other hand, when $q = 2$ we have $n \geq 5$ by our assumption; hence by Theorem 3.1 [6] $d(G) = 2^n - 2 = 2p$, a contradiction. (The desired contradiction in the case $|G|_p = p$ could also be derived from Theorem 6.1 [3].)

In the latter case it is easy to see that $\chi(1) = |G|_p$ divides $(q^n - 1)/(q - 1)$. First we consider the subcase $q \geq 3$. Clearly, D also divides $(q^n - 1)/(q - 1)$ (where D is as defined in Lemma 2.7). Due to Lemma 2.7, $D = k \cdot q^{n-1} + l$ for some integers $k \geq 1$ and $l = \pm 1$. Since $2q^{n-1} - 1 > (q^n - 1)/(q - 1)$, $k = 1$, i.e., $D = q^{n-1} \pm 1$. But

$$q^{n-1} + 1 < \frac{q^n - 1}{q - 1} < 2(q^{n-1} - 1);$$

therefore D cannot divide $(q^n - 1)/(q - 1)$, a contradiction. Finally, let $q = 2$. Then $n \geq 5$, and so by Theorem 3.1 [6], $G = SL_n(2)$ has no nontrivial irreducible characters of degree dividing $2^n - 1$, again a contradiction.

5. ORTHOGONAL GROUPS

In this section we prove Theorem 1.2 for the case of orthogonal groups. Let $q = r^m$ be a power of a prime r, $G = P\Omega_l^{\epsilon}(q)$ and $l \geq 7$, where $\epsilon = \pm$ if l is even and ϵ is void if l is odd. In the case l is even, $N \pm \epsilon$ is understood as $N \pm 1$ for $\epsilon = +$ and $N \mp 1$ for $\epsilon = -$.

Suppose G has a p-Steinberg character χ, where p divides $|G|$ but not q. First we establish the existence of a subgroup $R < G$ of extraspecial type which satisfies the conditions of Lemma 2.7.

PROPOSITION 5.1. *G contains a subgroup R of extraspecial type of order $q^{1+2(l-4)}$ with $|Z(R)| = q$, which satisfies the conditions of Lemma* 2.7. *If D is as defined in Lemma* 2.7, *then D divides*

$$\mathscr{Q}_2 = \left(\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{q^2 - 1} \right)_p$$

if $l = 2n + 1$ *is odd, and*

$$\mathcal{Q}_4 := \left(\frac{(q^{n-2} + \epsilon)(q^{2n-2} - 1)(q^n - \epsilon)}{q^2 - 1} \right)_p$$

if $l = 2n$ *is even.*

Proof. (1) First we consider $H = SO_l^{\epsilon}(q)$ and its natural module $V = \mathbb{F}_q^l$, endowed with a nondegenerate H-invariant quadratic form Q. Let $(\cdot, \cdot)$ denote the symmetric bilinear form associated to $Q: (x, y) = Q(x + y) - Q(x) - Q(y)$. Consider a basis $(v_1, \dots, v_l)$ of V, in which $(\cdot, \cdot)$ has Gram matrix of the form

$$\begin{pmatrix} 0 & E_2 & 0 \\ E_2 & 0 & 0 \\ 0 & 0 & J \end{pmatrix},$$

where $J = \mathrm{diag}(\alpha_1, \dots, \alpha_{l-4})$ if q is odd, and $J = \left(\begin{smallmatrix} 0 & E_{l/2-2} \\ E_{l/2-2} & 0 \end{smallmatrix}\right)$ if q is even. This is possible because $l > 4$.

Put $V_1 = \langle v_1\ v_2 \rangle_{\mathbb{F}_q}$, $V_2 = V_1^{\perp}$. The desired subgroup R is $O_r(P)$, where $P = St_H(\langle v_1, v_2 \rangle_{\mathbb{F}_q})$ is the stabilizer of an isotropic line. In other words, R consists of all elements of P that act trivially on the spaces V_1, V_2/V_1 and V/V_2. In the basis $(v_1, \dots, v_l)$ of V, any $f \in R$ is represented by the matrix

$$[X, Y] := \begin{pmatrix} E_2 & Y & -{}^tX \cdot J \\ 0 & E_2 & 0 \\ 0 & X & E_{l-4} \end{pmatrix}$$

for some $X \in M_{l-4,2}(\mathbb{F}_q)$, $Y \in M_{2,2}(\mathbb{F}_q)$ with

$$Y + {}^tY + {}^tXJX = 0. \tag{1}$$

If q is odd, then (1) is necessary and sufficient for f to belong to R. If ${}^tXJX = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, $a, b, c \in \mathbb{F}_q$, then (1) is equivalent to $Y = \left(\begin{smallmatrix} -a/2 & y \\ -b-y & -c/2 \end{smallmatrix}\right)$ for any $y \in \mathbb{F}_q$. Hence $|R| = q^{1+2(l-4)}$.

If q is even, then (1) together with the equation

$$Q(f(v_3)) = Q(v_3), \qquad Q(f(v_4)) = Q(v_4) \tag{2}$$

is necessary and sufficient for f to belong to R. It is easy to see that for any $X \in M_{l-4,2}(\mathbb{F}_q)$, ${}^tXJX = \left(\begin{smallmatrix} 0 & b \\ b & 0 \end{smallmatrix}\right)$ for a certain $b \in \mathbb{F}_q$. Hence (1) is equivalent to $Y = \left(\begin{smallmatrix} x & y \\ b+y & t \end{smallmatrix}\right)$ for any $x, y, t \in \mathbb{F}_q$. Furthermore, (2) is equivalent to $x = Q(u_3)$ and $t = Q(u_4)$, where u_3 and u_4 are the orthogonal projections of $f(v_3)$ and $f(v_4)$ onto $\langle v_5, \dots, v_l \rangle_{\mathbb{F}_q}$, respectively. Hence $|R| = q^{1+2(l-4)}$.

The group multiplication in R is given by $[X,Y]\circ[X',Y'] = [X+X', Y + Y' - {}^tXJX']$. Using this formula, it is easy to check that $[R,R] = Z(R)$ is equal to

$$T = \left\{ t_c := \left[0, \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}\right] \middle| c \in \mathbb{F}_q \right\}.$$

Moreover, $[f,R] = T$ for any $f \in R \setminus T$. Thus R is of extraspecial type. Also, if $h_A \in H$ has the matrix $\operatorname{diag}(A, {}^tA^{-1}, E_{l-4})$ (in the indicated basis of V), $A \in GL_2(q)$, then $h_A t_c h_A^{-1} = t_{c \cdot \det A}$. This means that all nontrivial elements $t_c \in T$ are conjugate in H.

(2) Next we consider $K = [H,H] = \Omega_l^{\epsilon}(q)$. It is known that K has index 2 in H. In particular, $h_{A^2} = h_A^2 \in K$ for any $A \in GL_2(q)$. Hence the above computation shows that $(t_c)^K \cap T$ has length $q-1$ or $(q-1)/2$ for $c \in \mathbb{F}_q^{\cdot}$. We also know that all $C_K(t_c)$ with $c \neq 0$ are H-conjugate.

If $q \geq 3$, then $[GL_2(q), GL_2(q)] = SL_2(q)$; hence $C_K(t_c)$ contains the subgroup

$$\begin{aligned}\left\{ h_{A,B} := \operatorname{diag}(A, {}^tA^{-1}, B) \,|\, A \in SL_2(q),\, B \in \Omega_{l-4}^{\epsilon}(q) \right\} \\ \simeq SL_2(q) \times \Omega_{l-4}^{\epsilon}(q).\end{aligned}$$

If $q = 2$, then $[GL_2(q), GL_2(q)] \simeq \mathbb{Z}_{q^2-1}$; hence $C_K(t_c)$ contains the subgroup

$$\left\{ \operatorname{diag}(A, {}^tA^{-1}, B) \,|\, A \in \mathbb{Z}_{q^2-1},\, B \in \Omega_{l-4}^{\epsilon}(q) \right\} \simeq \mathbb{Z}_{q^2-1} \times \Omega_{l-4}^{\epsilon}(q).$$

In any of these cases, we see that $(K : C_K(t_c))_p$ divides $(|SO_l^{\epsilon}(q)| / |SO_{l-4}^{\epsilon}(q)| \cdot (q^2-1))_p$, which is equal to $\mathcal{Q}_2$ if $l = 2n+1$ is odd, and

$$\left(\frac{(q^{n-2}+\epsilon)(q^{2n-2}-1)(q^n-\epsilon)}{q^2-1} \right)_p$$

if $l = 2n$ is even.

(3) Observe that $R < K$. (For, it is so if q is odd, since $|R|$ is then odd and $(H:K) = 2$. Now assume that q is even. As $T = [R,R], T$ is contained in K. If $q = 2$, then one can view R/T as a direct sum of two copies of the natural module $\mathbb{F}_2^{l-4}$ of $\Omega_{l-4}^{\epsilon}(2) < K$; hence the irreducibility of $\Omega_{l-4}^{\epsilon}(2)$ on this module implies that R is contained in K. Next assume $q \geq 4$. Then for $A \in GL_2(q)$, $B \in \Omega_{l-4}^{\epsilon}(q)$, we see that $h_{A,B} \in K$ and $h_{A,B} \cdot [X,Y] \cdot h_{A,B}^{-1} = [AY{}^tA, BX{}^tA]$. Specializing $A = aE_2$, $B = E_{l-4}$, $a \in \mathbb{F}_q^{\cdot}$, and forgetting the coordinate Y as $T < K$, we can say that $aX \in K$ whenever $X \in K$. Assume that $X \in R \setminus (R \cap K)$. Pick $a \in \mathbb{F}_q \setminus \{0,1\}$. Since $(H:K) = 2$, either $aX \in K$ or $(a-1)X \in K$. In the former case $X =$

$a^{-1}(aX) \in K$, a contradiction. In the latter case $X = (a-1)^{-1}((a-1)X) \in K$, again a contradiction.)

Moreover, $R \cap Z(K) = 1$ because $Z(K)$ has order dividing $\gcd(2, q-1)$. Therefore, we can view R as a subgroup of $G = K/Z(K)$. The discussion in parts (1), (2) of the present proof now ensures that R enjoys all the properties mentioned in Proposition 5.1. ∎

Next we consider the case $G = \Omega_{2n+1}(q)$, $n \geq 3$ and q an odd prime power. Then we can apply Proposition 5.1 (with $l = 2n+1$) and Lemma 2.7. On the one hand, the integer D defined in Lemma 2.7 is at least $q^{2n-2} - 1$ (since $C = 1$ would imply that Ker χ contains all the transvections t_c and so $\chi(1) = 1$, a contradiction). On the other hand, $D \leq \mathcal{Q}_2 < q^{2n-2} - 1$ by Lemma 2.4, a contradiction.

Finally, let $G = P\Omega^{\epsilon}_{2n}(q)$, $n \geq 4$. Applying Proposition 5.1 (with $l = 2n$) and Lemma 2.7, we see that the integer D defined in Lemma 2.7 is at least $q^{2n-3} - 1$. On the other hand, $D \leq \mathcal{Q}_4$. If $n = 4$, then one easily checks that $\mathcal{Q}_4 < q^5 - 1$, a contradiction. Assuming $n \geq 5$, we apply Lemma 2.4 and arrive at one of the following subcases.

(1) $\mathcal{Q}_2 < q^{n-1} - 1$. If $\epsilon = -1$ or if $\epsilon = 1$ but $q^{n-2} + \epsilon$ is not a p-power, then clearly

$$\mathcal{Q}_4 \leq \mathcal{Q}_2 \cdot \left(q^{n-2} - 1\right) < \left(q^{n-1} - 1\right)\left(q^{n-2} - 1\right) < q^{2n-3} - 1,$$

a contradiction. If $\epsilon = 1$ and $q^{n-2} + 1$ is a p-power, then by Lemma 2.2 either $(n, q, p) = (5, 2, 3)$, or $2 \mid q$ and $p = q^{n-2} + 1$. In the former case $\mathcal{Q}_4 = 9 < 2^7 - 1$, again a contradiction. In the latter case $\mathcal{Q}_4 = q^{n-2} + 1 < q^{2n-3} - 1$, a contradiction.

(2) $o_p(q) = n$ or $2n$. In this subcase p does not divide $q^{2n-4} - 1$ and $q^{2n-2} - 1$; hence $\mathcal{Q}_4 = (q^n - \epsilon)_p \leq q^n + 1 < q^{2n-3} - 1$, a contradiction.

(3) $2 \mid q$ and $p = q^{n-1} \pm 1$. In this subcase p does not divide $q^{2n-4} - 1$ and $q^{2n} - 1$; hence $\mathcal{Q}_4 = (q^{n-1} \pm 1)_p \leq q^{n-1} + 1 < q^{2n-3} - 1$, again a contradiction.

Thus we have proved Theorem 1.2 for the orthogonal groups.

6. EXCEPTIONAL GROUPS

Let G be a finite simple exceptional group of Lie type in characteristic r. Suppose G has a p-Steinberg character χ with p dividing $|G|$ and $p \neq r$.

First we exclude a few possibilities for (G, p).

LEMMA 6.1. *(G,p) cannot be any of the pairs $(F_4(3),2),({}^2E_6(3),2),(E_7(2),3)$.*

Proof. First we consider the case where $p=2$ and $G=F_4(3)$ or $G={}^2E_6(3)$. Observe that G contains a subgroup $H\simeq\Omega_9(3)$. Furthermore, by Proposition 5.1 H contains $R\simeq 3_{\pm}^{1+10}$. If $Z(R)=\langle t\rangle$ and D denotes $(G:C_G(t))_2$, then $D=2^m$ with $0\le m\le \nu_2(|G|)\le 19$. On the other hand, D^2-1 is divisible by 3^6, according to Lemma 2.7. By Lemma 2.1, $\nu_3(m)\ge 5$. Since $m\le 19$, we must have $D=1$, and so $t\in\operatorname{Ker}\chi$, a contradiction.

Next let $(G,p)=(E_7(2),3)$. Since G contains a subgroup $H\simeq 3\cdot{}^2E_6(2)$ and ${}^2E_6(2)>2_{\pm}^{1+20}$ (cf. [1]), we are convinced that $G>R\simeq 2_{\pm}^{1+20}$. If $Z(R)=\langle t\rangle$ and D denotes $(G:C_G(t))_3$, then $D=3^m$ with $0\le m\le \nu_3(|G|)=11$. On the other hand, D^2-1 is divisible by 2^{11}, according to Lemma 2.7. By Lemma 2.1, $\nu_2(m)\ge 8$. Since $m\le 11$, we must have $D=1$, and so $t\in\operatorname{Ker}\chi$, again a contradiction. ∎

Now we consider the exceptional groups case by case. The majority of the cases will be excluded by the reason that $|G|_p$ is strictly smaller than the Landázuri−Seitz−Zalesskii bound $l(G)$ (cf. [4, 5]). Where $|G|_p$ exceeds $l(G)$, one applies Lemma 6.1.

(1) $G=G_2(q)$. The cases $q=3,4$ can be directly excluded using [1]; hence we may suppose that $q\ge 5$. In particular, $d(G)\ge l(G)=q(q^2-1)$. If $p=2$, then by Lemma 2.1 $\chi(1)=|G|_2\le 4(q-\alpha)^2$ with $\alpha=\pm 1$ and $q\equiv\alpha \pmod 4$. If p is odd and $p\mid(q^2-1)$, then $|G|_p\le 3(q+1)^2$. If p divides q^6-1 but not q^2-1, then $|G|_p\le q^2+q+1$. In any of these cases we see that $1<\chi(1)<d(G)$, a contradiction.

(2) $G={}^2G_2(q)$. Here we have $q\ge 27$ and $l(G)=q(q-1)$. On the other hand, $|G|_p$ is at most $2(q+1)$ if $p=2$ and $q+\sqrt{3q}+1$ if $p>2$. Thus we obtain the desired contradiction that $1<\chi(1)<d(G)$.

(3) $G={}^2B_2(q)$. Using [1] we may suppose that $q>8$; hence $l(G)=(q-1)\sqrt{q/2}$. On the other hand, $|G|_p$ is at most $q+\sqrt{2q}+1$, and so $1<\chi(1)<d(G)$.

(4) $G={}^3D_4(q)$. The case $q=2$ can be directly excluded using [1]; hence we may suppose that $q\ge 3$. Recall that $l(G)=q^3(q^2-1)$. On the other hand, $|G|_p$ is at most $(q^2-1)^2$ if $p=2$, $9(q+1)^2$ if $2\ne p\mid(q^2-1)$, and $(q^2+q+1)^2$ if $p\nmid(q^2-1)$. In any of these cases $1<\chi(1)<d(G)$.

(5) $G={}^2F_4(q)$. The case $G={}^2F_4(2)'$ can be directly excluded using [1]; hence we may suppose that $q\ge 8$. Recall that $l(G)=q^4(q-1)\sqrt{q/2}$. On the other hand, $|G|_p$ is at most $(q-1)^2$ if $p\mid(q-1)$, $3(q+1)^2$ if $p\mid(q+1)$, $(q^2+1)^2$ if $p\mid(q^2+1)$, and q^4-q^2+1 otherwise. In any of these cases $1<\chi(1)<d(G)$.

(6) $G = F_4(q)$. The case $G = F_4(2)$ can be directly excluded using [1]; hence we may suppose that $q \geq 3$. Then $l(G) \geq q^6(q^2 - 1)$. On the other hand, $|G|_p$ is at most $2^7(q+1)^4$ if $p = 2$, $9(q+1)^4$ if $p = 3$, $(q+1)^4$ if $p \mid (q^2 - 1)$ and $p \neq 2, 3$, $(q^2+1)^2$ if $2 \neq p \mid (q^2+1)$, and $(q^2+q+1)^2$ otherwise. But $\chi(1) \geq d(G)$; hence we must have that $q = 3$ and $p = 2$. This final configuration is impossible due to Lemma 6.1.

(7) $G = E_6(q)$. Here $l(G) = q^9(q^2 - 1)$. Furthermore, when $p = 2$ by Lemma 2.1 we have $|G|_p \leq 2^7(q-1)^6$ if $q \equiv 1 \pmod 4$ and $|G|_p \leq 2^9(q+1)^4$ if $q \equiv 3 \pmod 4$. Similarly, when $p = 3$ we have $|G|_p \leq 3^4(q-1)^6$ if $q \equiv 1 \pmod 3$ and $|G|_p \leq 3^2(q+1)^4$ if $q \equiv 2 \pmod 3$. If $p \mid (q-1)$ and $p \geq 5$ then $|G|_p \leq 5(q-1)^6$. If $p \mid (q+1)$ and $p \geq 5$ then $|G|_p \leq (q+1)^4$. If $p \nmid (q^2 - 1)$ then $|G|_p \leq (q^2+q+1)^3$. In all cases $|G|_p < l(G)$.

(8) $G = {}^2E_6(q)$. The case $q = 2$ can be directly excluded using [1]; hence we may suppose that $q \geq 3$. Here $l(G) = q^9(q^2 - 1)$. Furthermore, when $p = 2$ we have $|G|_p \leq 2^7(q+1)^6$ if $q \equiv 3 \pmod 4$ and $|G|_p \leq 2^9(q-1)^4$ if $q \equiv 1 \pmod 4$. Similarly, when $p = 3$ we have $|G|_p \leq 3^4(q+1)^6$ if $q \equiv 2 \pmod 3$ and $|G|_p \leq 3^2(q-1)^4$ if $q \equiv 1 \pmod 3$. If $p \mid (q+1)$ and $p \geq 5$ then $|G|_p \leq 5(q+1)^6$. If $p \mid (q-1)$ and $p \geq 5$ then $|G|_p \leq (q-1)^4$. If $p \nmid (q^2 - 1)$ then $|G|_p \leq (q^3+1)^2$. In all cases $|G|_p < l(G)$. The only exception $(q, p) = (3, 2)$ is excluded by Lemma 6.1.

(9) $G = E_7(q)$. Here $l(G) = q^{15}(q^2 - 1)$. Furthermore, when $p = 2$ we have $|G|_p \leq 2^3((q^2-1)_2)^7$. Similarly, when $p = 3$ we have $|G|_p \leq 3^4((q^2-1)_3)^7$. If $p \mid (q^2 - 1)$ and $p \geq 5$ then $|G|_p \leq 7(q+1)^7$. If $p \nmid (q^2 - 1)$ then $|G|_p \leq q^9 + 1$. In all cases $|G|_p < l(G)$. The only exception $(q, p) = (2, 3)$ is excluded by Lemma 6.1.

(10) $G = E_8(q)$. Here $l(G) = q^{27}(q^2 - 1)$. Furthermore, when $p = 2$ we have $|G|_p \leq 2^6((q^2-1)_2)^8$. Similarly, when $p = 3$ we have $|G|_p \leq 3^5((q^2-1)_3)^8$. If $p \mid (q^2 - 1)$ and $p \geq 5$ then $|G|_p \leq 5^2(q+1)^8$. If $p \nmid (q^2 - 1)$ then $|G|_p \leq q^{15} + 1$. In all cases $|G|_p < l(G)$.

Thus we have proved Theorem 1.2 for the exceptional groups of Lie type.

7. SPORADIC GROUPS

Let G be any of the 26 sporadic finite simple groups and suppose that G has a p-Steinberg character χ. Using [1], one easily verifies that this cannot happen if $G \neq BM, F_1$ or if $p \neq 2$. Assume now that $p = 2$ and $G = BM$ or $G = F_1$. Then G contains $R = 3_+^{1+8}$. By Lemma 2.7, there exists m, $1 \leq m \leq \nu_1(|G|) \leq 46$ such that 3^5 divides $2^{2m} - 1$. This means by Lemma 2.1 that $\nu_3(m) \geq 4$, yielding $m \geq 81$, a contraction.

Theorem 1.2 has completely been proved.

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